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Investigation of Limit Cycle Oscillations Using Aeroelastic-Harmonic Balance Method

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This work investigates limit cycle oscillations in the transonic regime. A novel approach to predict Limit Cycle Oscillations using high fidelity analysis is exploited to accelerate calculations. The method used is an Aeroelastic Harmonic Balance approach, which has been proven to be efficient and able to predict periodic phenomena. The behaviour of limit cycle oscillations is analysed using uncertainty quantification tools based on polynomial chaos expansions. To improve the efficiency of the sampling process for the polynomial-chaos expansions an adaptive sampling procedure is used. These methods are exercised using two problems: a pitch/plunge aerofoil and a delta-wing. Results indicate that Mach n. variability is determinant to the amplitude of the LCO for the 2D test case, whereas for the wing case analysed here, variability in the Mach n. has an almost negligible influence in amplitude variation and the LCO frequency variability has an almost linear relation with Mach number. Further test cases are required to understand the generality of these results.

Nomenclature

Latin Symbols

A	Harmonic Balance frequency domain matrix
b, c	aerofoil semi-chord and chord, respectively
CFL	Courant-Friedrichs-Lewy
D	Harmonic Balance operator matrix
E	energy
E	Transformation matrix between frequency and time domains
f	fluid force acting on structure
F, G, H	convective fluxes for fluid equations
h	plunge coordinate
I	HB residual
K	structure stiffness matrix
L	frequency updating figure of merit
M	structure mass matrix
p	pressure
R	vector of fluid and/or structural equation residual
t	time step
U_∞	free-stream velocity
u, v, w	fluid cartesian velocity components
V, V_s	reduced velocity and velocity index
W	vector of fluid unknowns
x, y	vector of structural unknowns

Greek Symbols

α	angle of attack
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ω, κ frequency and reduced frequency, $\kappa = \frac{2\omega}{U_\infty c}$
 ρ density
 τ pseudo-time step

I. Introduction

Industry standard practices to solve aeroelastic problems rely heavily upon linear aerodynamic theory. This has well known limitations in the transonic regime and where other sources of aerodynamic nonlinearities are present (e.g., unsteady viscous flows), hence a clear need for physics based modelling tools has emerged as identified by Noll *et al.*¹ When nonlinearities are present, aeroelastic instabilities can lead to oscillations that become limited and limit cycle oscillations are observed. This is a problem of considerable practical interest and is well documented for in-service aircraft.^{2,3} The presence of nonlinearities, either structural or aerodynamic, poses additional challenges both in terms of complexity and computational resources, by requiring higher-fidelity analysis. Hence, several efforts have been made to address both issues of retaining the required level of fidelity to capture the relevant physics, while at the same time limiting the computational resources required for such analysis.

Advances in CFD (Computational Fluid Dynamics) methods allowed the coupling of nonlinear aerodynamic models with CSD (Computational Structural Dynamics) in the time domain; however this type of analysis is used as a last resort tool due to the high computational cost. To circumvent the need for expensive simulations, several kinds of Reduced Order Models (ROM) have been proposed and used, for example: Proper Orthogonal Decomposition (POD),^{4,5} Volterra Series,⁶⁻⁸ Neural Networks.⁹

An alternative to ROM and full time domain analysis of aeroelastic oscillatory problems is to employ the non-linear Harmonic Balance (HB) method. New Harmonic Balance methods have been developed for CFD time periodic flows;^{10,11} in such methods, the periodicity of the flow is exploited and represent time dependent flow variables as Fourier series and recast the problem in terms of Fourier coefficients. These methods have been successful in predicting unsteady flows efficiently in diverse applications: forced motions,^{12,13} helicopter rotors,¹⁴ turbomachinery.^{10,15,16} Thomas *et al.* extended the HB formulation to predict Limit Cycle Oscillations for fixed wing aircraft.³ Ekici and Hall further reduced the computational cost of predicting LCOs with HB methods, by proposing a *one-shot* method to analyze 1-DOF LCO in turbomachinery flows.¹⁵ Recently the authors extended this approach for fixed wing LCO computations.¹⁷ In this paper, this last variation of the HB method for LCO predictions is formulated around an implicit method originally proposed in ref.,¹² yielding a faster, more robust approach to nonlinear aeroelastic problems such as LCOs.

II. Flow Solver

The semi-discrete form of an arbitrary system of conservation laws such as the three-dimensional Euler equations can be described as:

$$\frac{\partial \mathbf{W}}{\partial t} = -\mathbf{R}(\mathbf{W}) \quad (1)$$

where \mathbf{R} is the residual error of the steady-state solution:

$$\mathbf{R} = \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} + \frac{\partial \mathbf{H}}{\partial z} \quad (2)$$

Here \mathbf{W} is the vector containing the flow variables and \mathbf{F} , \mathbf{G} , \mathbf{H} are the fluxes, which are given by:

$$\mathbf{W} = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho E \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ u(\rho E + p) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ v(\rho E + p) \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \rho w \\ \rho vw \\ \rho vw \\ \rho w^2 + p \\ w(\rho E + p) \end{bmatrix}, \quad (3)$$

The steady state solution of the Euler equations is obtained by marching the solution forward in time by solving the following discrete nonlinear system of equations:

$$\frac{\mathbf{W}^{n+1} - \mathbf{W}^n}{\Delta t} = -\mathbf{R}^n \quad (4)$$

To discretize the residual convective terms a Roe flux function¹⁸ together with MUSCL interpolation is used,¹⁹ the Van Albada limiter is used to obtain 2^{nd} order accuracy.

III. Harmonic Balance Formulation

As discussed in the introduction, several authors have demonstrated the suitability of HB methods as an alternative to time marching CFD formulations for periodic flow problems. To obtain the HB version of the flow solver, we follow the methodology proposed by Woodgate¹² for implicit Harmonic Balance methods, which is summarised next. Consider the semidiscrete form as a system of ordinary differential equations

$$\mathbf{I}(t) = \frac{d\mathbf{W}(t)}{dt} + \mathbf{R}(t) = 0 \quad (5)$$

The solution of \mathbf{W} and \mathbf{R} in eq.(5) can be approximated to be a truncated Fourier series of N_H harmonics with a fundamental frequency ω :

$$\mathbf{W}(t) \approx \hat{\mathbf{W}}_0 + \sum_{n=1}^{N_H} (\hat{\mathbf{W}}_{2n-1} \cos(n\omega t) + \hat{\mathbf{W}}_{2n} \sin(n\omega t)) \quad (6)$$

$$\mathbf{R}(t) \approx \hat{\mathbf{R}}_0 + \sum_{n=1}^{N_H} (\hat{\mathbf{R}}_{2n-1} \cos(n\omega t) + \hat{\mathbf{R}}_{2n} \sin(n\omega t)) \quad (7)$$

Hence, eq.(5) can also be approximated by a truncated Fourier series,

$$\mathbf{I}(t) \approx \hat{\mathbf{I}}_0 + \sum_{n=1}^{N_H} (\hat{\mathbf{I}}_{2n-1} \cos(n\omega t) + \hat{\mathbf{I}}_{2n} \sin(n\omega t)) \quad (8)$$

which results in the following system of equations

$$\hat{\mathbf{I}}_0 = \hat{\mathbf{R}}_0 \quad (9)$$

$$\hat{\mathbf{I}}_{2n-1} = \omega n \hat{\mathbf{W}}_{2n} + \hat{\mathbf{R}}_{2n-1} \quad (10)$$

$$\hat{\mathbf{I}}_{2n} = -\omega n \hat{\mathbf{W}}_{2n-1} + \hat{\mathbf{R}}_{2n} \quad (11)$$

which results in a system of $(2N_H + 1)$ equations for the Fourier coefficients that can be expressed in matrix form as

$$\omega \mathbf{A} \hat{\mathbf{W}} + \hat{\mathbf{R}} = 0 \quad (12)$$

where \mathbf{A} is given by:

$$\mathbf{A} = \begin{bmatrix} 0 & & & \\ & \mathbf{J}_1 & & \\ & & \ddots & \\ & & & \mathbf{J}_{N_H} \end{bmatrix}_{(2N_H+1) \times (2N_H+1)}, \quad \mathbf{J}_n = n \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad n = 1, 2, \dots, N_H \quad (13)$$

To overcome the difficulties in expressing the Fourier coefficient in $\hat{\mathbf{R}}$ as functions of $\hat{\mathbf{W}}$, Hall *et al.*¹⁰ proposed to cast the system of equations back in the time domain, where the flow variables and residual solutions are split into $(2N_H + 1)$, discrete, equally spaced intervals over the period $T = \frac{2\pi}{\omega}$.

$$\mathbf{W}_{hb} = \begin{bmatrix} \mathbf{W}(t_0 + \Delta t) \\ \mathbf{W}(t_0 + 2\Delta t) \\ \vdots \\ \mathbf{W}(t_0 + T) \end{bmatrix}, \quad \mathbf{R}_{hb} = \begin{bmatrix} \mathbf{R}(t_0 + \Delta t) \\ \mathbf{R}(t_0 + 2\Delta t) \\ \vdots \\ \mathbf{R}(t_0 + T) \end{bmatrix}, \quad (14)$$

It is possible to define a transformation matrix, \mathbf{E} that relates the frequency domain variables to their HB time domain counterpart¹⁰

$$\hat{\mathbf{W}} = \mathbf{E}\mathbf{W}_{hb} \quad \hat{\mathbf{R}} = \mathbf{E}\mathbf{R}_{hb} \quad (15)$$

Substituting the terms in eq.(15) in eq.(12), it becomes:

$$\begin{aligned} \omega \mathbf{A} \hat{\mathbf{W}} + \hat{\mathbf{R}} &= 0 = \omega \mathbf{A} \mathbf{E} \mathbf{W}_{hb} + \mathbf{E} \mathbf{R}_{hb} = \omega \mathbf{E}^{-1} \mathbf{A} \mathbf{E} \mathbf{W}_{hb} + \mathbf{R}_{hb} = \\ &= \omega \mathbf{D} \mathbf{W}_{hb} + \mathbf{R}_{hb} = 0 \end{aligned} \quad (16)$$

where $\mathbf{D} = \mathbf{E}^{-1} \mathbf{A} \mathbf{E}$, the elements in matrix \mathbf{D} are given by:

$$\mathbf{D}_{i,j} = \frac{2}{2N_H + 1} \sum_{k=1}^{N_H} k \sin \left(\frac{2\pi k(j-i)}{2N_H + 1} \right) \quad (17)$$

To solve eq.(16) a pseudo time step of the form is introduced:

$$\frac{d\mathbf{W}_{hb}}{d\tau} + \omega \mathbf{D} \mathbf{W}_{hb} + \mathbf{R}_{hb} = 0 \quad (18)$$

To solve eq.(18), any steady-state CFD time marching method can be used. In this work, a 4th - order Runge-Kutta method is used.

IV. Aeroelastic Formulation

Consider a generic dynamic system without damping, whose behaviour can be described using the equations of motion given by:

$$\mathbf{M} \ddot{\mathbf{x}} + \mathbf{K} \mathbf{x} = \mathbf{f} \quad (19)$$

where \mathbf{M} , \mathbf{K} , respectively, represent the mass and stiffness of the system and \mathbf{f} is an external force (in this work, this will be the aerodynamic force, $\mathbf{f} = \mathbf{f}(\mathbf{w}, \omega, \mathbf{x})$). This equation can be transformed into a state-space form, giving:

$$\dot{\mathbf{y}} = \mathbf{A}_s \mathbf{y} + \mathbf{B}_s \mathbf{f} \quad (20)$$

where:

$$\mathbf{A}_s = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{M}^{-1} \mathbf{K} & 0 \end{bmatrix}, \quad \mathbf{B}_s = \begin{bmatrix} 0 \\ \mathbf{M}^{-1} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} \quad (21)$$

Equation (20) has a similar form to the flow equations, hence it can be solved using the Harmonic Balance method described in the previous section, resulting in the following HB format of eq.(20):

$$\omega \mathbf{D} \mathbf{y}_{hb} = \mathbf{A}_s \mathbf{y}_{hb} + \mathbf{B}_s \mathbf{f}_{hb} \quad (22)$$

where \mathbf{D} is the same HB operator described in eq.(17). Equation (22) can be solved using the same pseudo time technique previously presented, leading to the following system of equations:¹⁵

$$\frac{d\mathbf{y}_{hb}}{d\tau} + \omega \mathbf{D} \mathbf{y}_{hb} - (\mathbf{A}_s \mathbf{y}_{hb} + \mathbf{B}_s \mathbf{f}_{hb}) = 0 \quad (23)$$

Equation (18) together with eq.(23) represent the nonlinear coupled aeroelastic system; when solving the aeroelastic system of equations, at each iteration, the generalized aerodynamic forces are computed using eq.(18), which will feed into eq.(23). The solution from eq.(23) will provide new generalized displacement and velocities to eq.(18). The CFD grid is deformed using Transfinite Interpolation and the mesh velocities are approximated by finite-differences⁹.

IV.A. Prediction of Limit-Cycle Oscillations

The prediction of LCO depends on determining a solution vector for $[\omega, \mathbf{y}]$ (the subscript *hb* is dropped for simplicity), that satisfies both the structural governing equation eq.(23) and eq.(18). If the LCO frequency, ω , is given beforehand, then the coupling itself becomes a fixed point iteration process which is extensively used for static aeroelastic problems in its time domain counterpart.¹³ Inspired by the results of Blanc *et al.*,¹³ the Yao and Marques¹⁷ proposed to transform this LCO prediction problem into a fixed point algorithm with frequency updating. To determine the LCO condition using eq.23, the frequency updating can be achieved by minimizing the L_2 norm of the residual \mathbf{R} of eq.(23).¹⁵ First, define a figure of merit, in this case:

$$\mathbf{L}_n = \frac{1}{2} \mathbf{R}^T \mathbf{R} = \frac{1}{2} [\omega \mathbf{D} \mathbf{y} - (\mathbf{A}_s \mathbf{y} + \mathbf{B}_s \mathbf{f})]^T [\omega \mathbf{D} \mathbf{y} - (\mathbf{A}_s \mathbf{y} + \mathbf{B}_s \mathbf{f})] \quad (24)$$

The frequency is updated by minimizing the residual \mathbf{R} but, critically, without freezing the aerodynamic forces \mathbf{f} , leading to:

$$\frac{\partial \mathbf{L}_n}{\partial \omega} = \left(\mathbf{D} \mathbf{y} - \mathbf{B}_s \frac{\partial \mathbf{f}}{\partial \omega} \right)^T [\omega \mathbf{D} \mathbf{y} - (\mathbf{A}_s \mathbf{y} + \mathbf{B}_s \mathbf{f})] \quad (25)$$

If the frequency ω is not at the LCO condition, the residual \mathbf{R} for the displacement is not able to converge. Therefore, the idea is to update the frequency at every n_i iterations. The full details of the algorithm are given in.¹⁷ When compared to the standard fixed point algorithm described by Blanc *et al.*,¹³ the new algorithm introduces some extra computational effort to compute the gradient of the aerodynamic force with respect to the frequency. However, the frequency is only updated every n_i iterations (enough to reduce the residual by three orders of magnitude and allow an accurate estimation of eq.(25), typically every 10-15 iterations) and the perturbation is sufficiently small, minimizing the computational cost.

V. Results

V.A. Pitch/plunge aerofoil

To exercise the method, a two degree-of-freedom, pitch/plunge NACA 64_A010 aerofoil is used. The transonic flow, ($M_\infty = 0.8$) is considered to be inviscid. A validation and grid convergence study have been performed reported in reference ?? and a 4000 point grid as shown in fig.V.A was used. The equations of motion are described in eq.26

$$m \ddot{h} + S_\alpha \ddot{\alpha} + k_h h = -q_\infty c C_l \quad (26)$$

$$S_\alpha \ddot{h} + I_\alpha \ddot{\alpha} + k_\alpha \alpha = -q_\infty c^2 C_m \quad (27)$$

The flutter condition is $[\omega, V_s] = [0.1089, 0.693]$ Following Thomas *et al.*²⁰ the non-dimensional form of eq.(19) for this problem becomes:

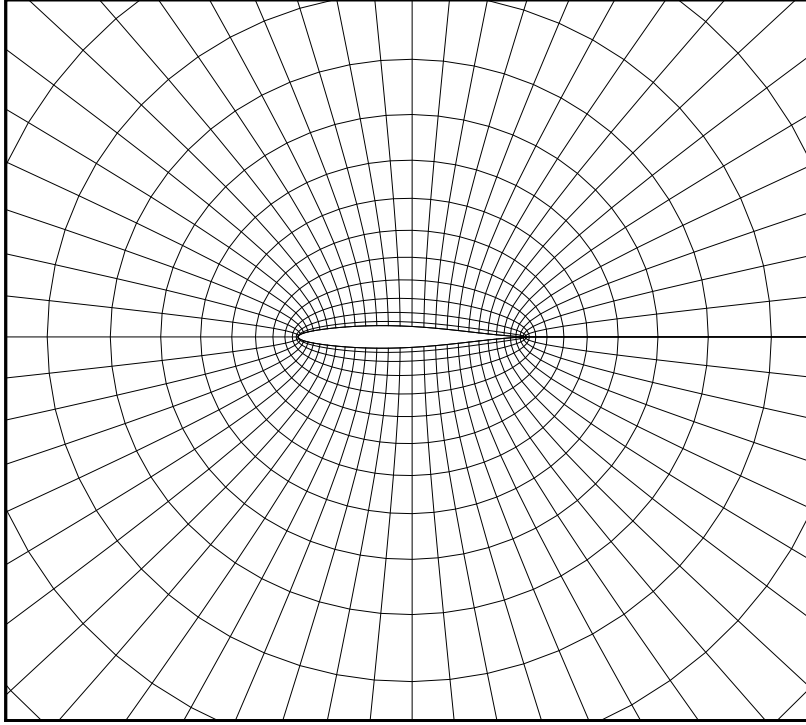
$$\mathbf{M} \ddot{\mathbf{y}} + \frac{1}{V^2} \mathbf{K} \mathbf{y} = \frac{4}{\pi \mu} \mathbf{f} \quad (28)$$

the pitch-plunge aerofoil structural parameters are given by:

$$\mathbf{M} = \begin{bmatrix} 1 & x_\alpha \\ x_\alpha & r_\alpha^2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \left(\frac{\omega_h}{\omega_\alpha} \right)^2 & 0 \\ 0 & r_\alpha^2 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} -C_l \\ 2C_m \end{bmatrix},$$

$$\mathbf{y} = \begin{bmatrix} \frac{h}{b} \\ \alpha \end{bmatrix}, \quad V = \frac{U_\infty}{\sqrt{\mu \omega_\alpha b}}$$

with the remainder parameters given in table 1. The plunge direction is represented by h and pitch by α with the respective frequencies, ω_h and ω_α , S_α, I_α are the first and second moments of inertia of the aerofoil about the elastic axis, m is the structure's mass and b is the half chord, V and U_∞ are the reduced and original free stream velocities, respectively.



In addition to the above parameters, the Mach number is set to 0.8 and all calculations are started at an angle of attack of 0° ; the aeroelastic axis distance from the centre chord is set to zero. A variation in pitch and plunge of 0.01° and $0.01b$ are used to perturb the system away from its initial position. The flutter conditions are obtained from ref.,²¹ limit-cycle oscillations are reported to occur at $[V, \mu] = [44, 4400]$. Figure ??-(a) shows the convergence of the fluid HB system, when the fluid system residual is reduced by eight orders of magnitude, the frequency is recomputed following eq.(25) to drive the structural residual to convergence, as the structural system is updated, the residual reduces following a *staircase* pattern, similar to the convergence of the frequency of oscillations.

VI. Conclusions & Outlook

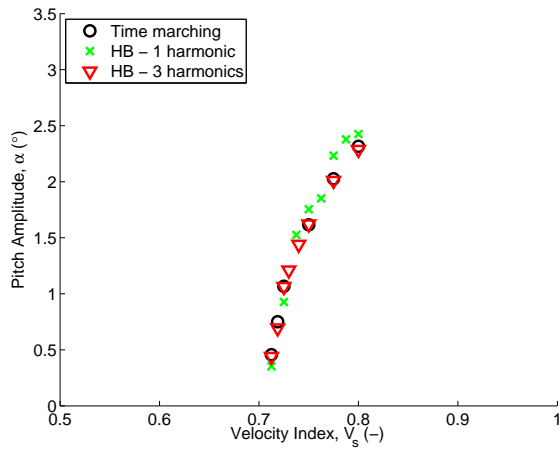
The implementation of a framework to predict limit cycle oscillations was coupled with an implicit HB CFD solver. The application of implicit methods provides faster convergence of the CFD residual by allowing the use of larger CFL numbers, which was considered a bottleneck for this approach. Initial results for a pitch/plunge aerofoil are encouraging and show the potential of this approach to predict nonlinear, periodic aeroelastic instabilities for large and more complex cases.

Acknowledgments

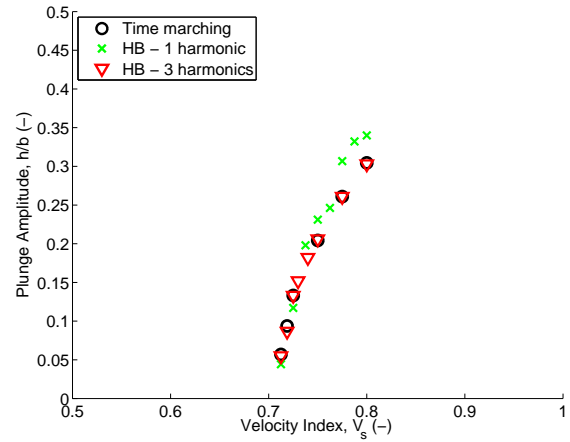
This work was sponsored by the United Kingdom Engineering and Physical Sciences Research Council (grant number EP/K005863/1). The authors gratefully acknowledged this support.

Static unbalance, $x_\alpha = S_\alpha/mb$	0.25
Radius of gyration about elastic axis, $r_\alpha^2 = I_\alpha/mb^2$	0.75
Frequency ratio, ω_h/ω_α	0.5
Mass ratio, $\mu = m/\pi\rho_\infty b^2$	75

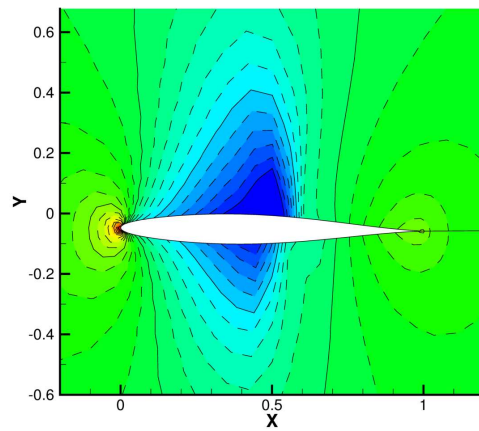
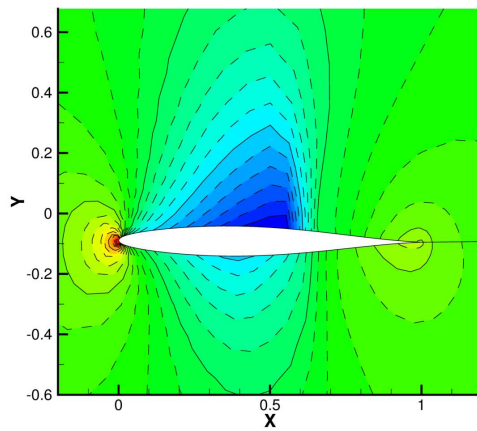
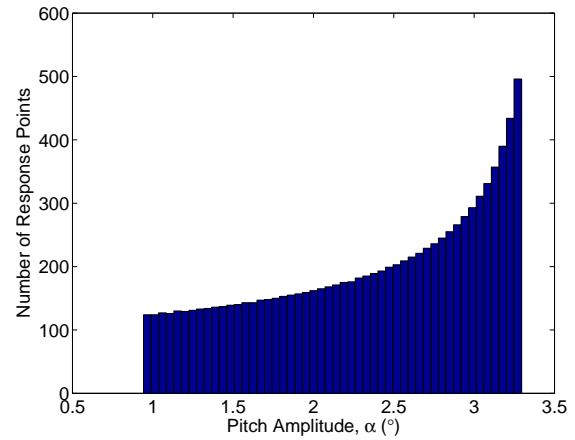
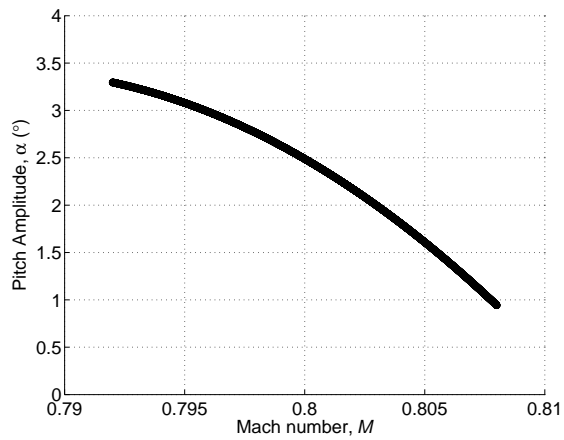
Table 1. Pitch/Plunge Aerofoil Parameters

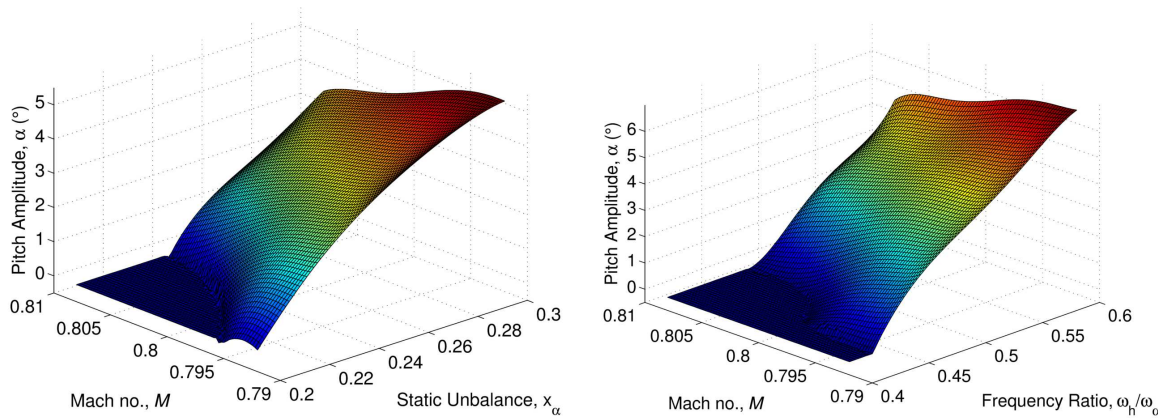


$V_s = 0.775$



$V_s = 0.80$





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